Swami Ramanand Teerth Marathwada University, Nanded, Maharashtra, India.

*(UGC recognized, NAAC Re-accredited **'B++'** grade state university)*

A Study of Waves and Oscillations and It's Applications using Fractional Calculus

Research proposal submitted
In partial fulfillment of registration to
Ph.D Programme

In The Subject

MATHEMATICS

Under Faculty of

Science and Technology

Submitted by

Miss. Mangrulkar Bhagyashri Umesh

M.Sc(Mathematics), SET

Email: mangrulkar.b@gmail.com

Under the Supervision of

Dr. Muley Yogesh Manohar

Asst. Prof and Head of Dept. of Mathematics

Kai. Rasika Mahavidyalaya, Deoni-413519.

Email: yogesh.m.muley@gmail.com

Year 2022

A Study of Waves and Oscillations and It's Applications using Fractional Calculus

1. Introduction to proposed research topic:

In every life we come across numerous things that move. These motions are of two types

i) The motion in which the body moves about a fixed point

ii) The motion in which the body moves from one place to other w.r.t time. The first type of motion of a body about a fixed pt is called oscillatory motion.

Wave motion is a form of disturbance which travels through the medium due to the repeated periodic motion of the particles of the medium about their mean positions, the disturbance being banded over from one particle to the next. It is essential to understand the concept of wave motion in the study of various branches in physics. Wave motion refers to the transfer of energy from one point to another point of the medium.

These phenomena attracted mathematicians to explore them with the tools of fractional calculus. Simple solutions using FC techniques were presented by Kulish & Lage. [1] Attempts have been made to study non-conservative systems using Lagrangian containing derivatives of fractional order.

Yuan and Agrawal [2]presented a numerical scheme for solving a fractionally damped spring-mass damper system with one degree of freedom using the Laguerre integral formula and considering the derivative of order 0.5 in their example. A comparison of numerical methods is found in Shokooh and Suarez. [3] They proposed a closed form solution to the problem using an eigen vector expansion method with fractional derivative of order 0.5 only. Laplace transform was used by Bagley and Trovik. [4] Fourier transform was used by Gaul et al [5] with fractional order of 0.5 only. Hartley and Lorenzo [6] employed the Mittag-Leffler function to study the initialization problem for a system of linear fractional order differential equations for discrete values of the fractional order of derivation. Koeller [7] studied creep and relaxation functions taking the order of derivative to vary from 0.05 to 0.35 to show that a continuous transition from solid state to the fluid state occurs when the memory parameter varies from zero to one expressing the results in terms of Mittag-Liffler function. Diethelm et al [8]offered a useful discussion and comparison of different numerical solutions for fractional equations.

- ✓ The diffusion equations describe a process, where a disturbance of the initial conditions spreads infinitely fast, the propogation velocity of the disturbance is constant for the wave equation. In a certain sense, the time-fractional diffusion- wave equation that is obtained from the diffusion equation by substituting the first derivative in time by the fractional derivative of order $\alpha, 1 < \alpha < 2$, interpolates between these two different behaviors. Namely, whereas a response of its fundamental solution to a localized disturbance spreads infinitely fast, its maximum location disperses with a finite velocity $v(t, \alpha)$ that is determined by the formula [9]

$$v(t, \alpha) = C_\alpha t^{\frac{\alpha}{2}-1}$$

For $\alpha = 1$(diffusion), the propogation velocity is equal to zero because of $C_1 = 0$

For $\alpha = 2$(Wave propogation) it remains constant, whereas for all intermediate values of α the propogation velocity of the maximum point depends on time t and is a decreasing function that varies from $+\infty$ at time t=0+ to zero as $t \to +\infty$. This fact makes it difficult to interpret solutions to the time-fractional diffusion wave equations.

- ✓ The harmonic oscillator, given by a linear differential equation of second order with constant coefficients is a cornerstone of the classical mechanics. Today, this elementary(and fundamental) conception has the widest origin of physical, chemical, engineering applications and needs no introduction. Its success mainly rests on its universality, and its simplicity gives boundless intrinsic capabilities for sweeping generalization. Suffice it to recall the passage from the language of functions in phase space to operators in Hilbert space so that the oscillatory model came strongly in the quantum theory. Therefore, the fractional calculus has made also an important contribution to this way.

❖ Some Basic information about Fractional Calculus:

The idea of Fractional Calculus (FC) was put forward by a question asked by L' Hospital to Leibnitz [10]. His question was, "What is the meaning of $\left(\frac{d^n y}{dx^n}\right)$ if n= $\frac{1}{2}$?" Leibnitz replied that, "It will lead to a paradox." But he added " From this apparent paradox, one day useful consequences will be drawn."

This conversion between these two legends led to a new era of fractional calculus. Further interest was taken by Lagrange in 1772. Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Weyl in 1919 & many more. [11] In 1819 Lacroix [12] answered to the problem created by Leibniz & L' Hospital.

He claimed that $\frac{d^{1/2}x}{dx^{1/2}} = 2\sqrt{\frac{x}{\pi}}$

He developed a formula for nth derivative of $y = x^m \; ; x \in I^+$

$$\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n} \;\; ; m \geq n$$

Replacing the factorial symbol by Gamma function he developed the formula for the fractional derivative of a power function

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$$

where α &β are fractional numbers where the gamma function is defined by

$$\Gamma z = \int_0^{\infty} e^{-x} \; x^{z-1} dx \; ; z > 0$$

In 1823, Abel [13] considered the solution of the integral equation related to the tautochrone problem. He found that the solution could be accomplished via an integral transform which could be written as a semi-derivative.

The integral transform considered by Abel was

$$K = \int_0^{\infty} (x-t)^{-\frac{1}{2}} f(t) dt$$

Abel wrote the R.H.S of above equation by fraction as derivative of order ½ i.e.

$$K = \sqrt{\pi}\left(\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} f(x)\right)$$

Abel's solution had attracted the attention of J. Liouville, who made the first major study of fractional calculus. [14]

In 1832 he began to study fractional calculus & applied his obtained results to problems in potential theory. [15]

Liouville proposed the following expression for evaluating the derivative of order α

$$D_x^{\alpha} f(x) = \sum_{n=0}^{\infty} c_n a_n^{\alpha} e^{a_n x}$$

This is known as Liouville's first formula for a fractional derivative. [15]

But because of its relatively narrow scope, this formula can't be seen as a general definition of fractional derivatives so be worked hard & came upto the following result.

$$D_x^{\alpha}\, x^{\beta} \;=\; (-1)^{\alpha} \;\frac{\Gamma(\alpha+\beta)}{\Gamma\beta}\; x^{-\alpha-\beta} \;\; ;\, \beta > 0$$

This is known as Liouville's second formula.

And the further study was made by different mathematicians.

Reimann Liouville Fractional Derivative [16]:

For $\alpha \in [n-1, n)$, the α −derivatives of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{\alpha}^{t} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx$$

Caputo's Fractional derivative [16]:

For $\alpha \in [n-1, n)$, the α −derivatives of f is

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{\alpha} \frac{f^{(n)}(x)}{(t-x)^{(\alpha-n+1)}} dx$$

Mittag-Leffler function [17]**:**

The Mittag-Leffler function is an entire function defined by the series

$$E_{\alpha}(z) = \sum\nolimits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \;\; for\; \alpha > 0$$

- **Some Important Methods:**
  i. Laplace transform of Riemann Liouville Fractional Derivative [16]

$$L\left\{ {}_0 D_t^p f(t); s \right\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0 D_t^{p-k-1} f(t) \right]_{t=0}, (n-1 \le p < n)$$

  ii. Laplace transform of Caputo Fractional derivatives [16]

$$L\left\{ {}_0^C D_t^p f(t) \right\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0), \qquad (n-1 < p \le n), p > 0$$

2. Review of literature and development in the subject area:

W. Wyss(1986) [18]**:** In this literature the author obtained the fractional diffusion equation n with the help of Nigmatullin for a medium with fractal geometry(porous medium). He replaced the first time derivatives by λ-fractional time derivative($0 < \lambda \le 1$). He gave a solution in closed form in terms of Fox functions.

F. Mainardi(1996) [19]**:** In this literature, the author analysed the fractional relaxation-oscillation and Fractional diffusion- wave phenomena. He separately onbtained results for fractional relaxation(fractional oscillation) and fractional diffusion(fractional wave propogation). He obtained normalized spectrum of relaxation.

R.Hilfer(2000) [20]**:** In this literature, the author obtained the generalized fractional diffusion equation and stated its relation with continuous time random walk. He showed that the fractional diffusion is equivalent to a continuous time random walk whose waiting time density is a generalized Mittag-Leffler function.

Y. E. Ryabov and A. Puzenko(2002) [21]**:** On the basis of Mittag-Leffler function, the author has obtained the exact solution for fractional oscillator equation.The author led us to the conclusion that introducing a fractional time derivative of the Riemann-Lioville type into the

oscillator equation is qualitatively equivalent to introducing a dissipative term. The author has established a relation between the order of the fractional time derivative and the dissipative constant.

A. Tofighi(2003) [22]**:** In this literature, the author abtained the analytical expression for the time rate of change of potential energy, kinetic energy and total energy of a fractional oscillator in terms of the products of Mittag-Leffler functions. He gave a definition for the intrinsic damping force for fractional oscillator.

A. A. Stanislavsky(2004) [23]**:** In this literature, he had shown that the fractional oscillator can be considered as a model of the harmonic oscillator's medium. He analysed the features of media described by a fractional oscillator and an ensemble of ordinary harmonic oscillators with damping.

Al-rabtah, V.S. Erturk, S.Momani(2010) [24]**:** In this literature, they presented an algorithm for solving a fractional oscillator using the differential transform method. The fractional derivatives were described in the Caputo sense and the application of differential transform method was extended to obtain explicit and numerical solutions of linear and non-linear fractional oscillators.

Jun-Sheng Duan(2013) [25]**:** In this literature, the author has obtained the periodic solution of fractional oscillation equation with periodic input and arrived at the result- the periodic solution can be regarded as an asymptotic steady-state solution for a fractional oscillation with initial conditions.

Ming Li(2018) [26]**:** In this literature, the author addresses three classes of fractional oscillators named Class I,II,III. We know that the solutions to fractional oscillators of Class I type are represented by the Mittag-Leffler functions. But the closed form solutions to fractional oscillators in Classes II and III are unknown. The author has presented a theory of equivalent systems with respect to three classes of fractional oscillators.

Ji-Huan He(2019) [27]**:** In this literature, the author gives a fast estimation of periodic property of a non-linear oscillators, including He's frequency formulation, the max-min approach and the homotopy perturbation method.

G.M. Ismail, H.R. Abdl-Rahim(2019) [28]**:** In this literature, the authors gave an analytic exact solution for the fractional differential equation of the oscillator in a resisting medium via the natural transform method. The fractional derivatives were described in the Caputo sense.

Roman Provik(2020) [29]**:** In this literature, a numerical analysis of the Cauchy problem for a model class of fractional linear oscillators was carried out and forced oscillations of linear fractional oscillators are investigated

Zhi Yan, Xianbin Liu(2021) [30]**:** In this literature, the authors investigated the non-linear response at arbitrary order harmonic resonance and reported in a low frequency parameter driven fractional oscillator which endures bi-frequency external excitation and contains a distributed delay.

Qiubao Wang, Hao Wu(2022) [31]**:** In this literature, the author proposes the stiffness non linearities and asummetric smooth and discontinuous oscillators under time-delay feedback control with a fractional damping.

3. Objectives of Proposed research:

1. To study basic concepts of Fractional Calculus.
2. To study some methods to solve fractional differential equations.
3. To study fractional wave equations and fractional oscillators.
4. To compare the solutions of fractional diffusion equations and fractional oscillators.
5. To study recent applications of fractional wave equations and fractional oscillators.
6. To prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in physical and engineering field.

7. Hence, in the present work we will try to develop the new methods and solutions to fractional wave equations and fractional oscillators. The solutions will be represented graphically by Mathematica, MATLAB, Python, Scilab, Mathcad.

4. Research Methodology:

The following steps will be including in the methodology:

1. To refer the relevant reference books on fractional differential equations.
2. To study Mathematica, MATLAB, Python, Scilab, Mathcad softwares.
3. To study fractional diffusion equations and their various solutions.
4. To study fractional oscillators.
5. To represent the obtained solutions graphically with the help of mathematical softwares.

5. Importance of study:

In Physics and Engineering fractional calculus is becoming a valuable and important tool for analysing scientific problems. Fractional Calculus, which involves the integration and differentiation of an arbitrary or fractional order, provides additional tools that extend the descriptive capacity of calculus beyond the classical integer order concepts of rate of change and area under the curve.

- **International Status:** Many mathematicians and researchers across the world have contributed in developing fractional diffusion and wave equations. Some of them are: Leibnitz, Euler, L'Hospital, Lagrange, Laplace, Fourier, Abel, Lioville, Grunwald, Reimann, Riesz, Miller, Ross, Igor Podlubny, Mittag, Leffler, R. Hilfer, M. Caputo.
- **National Status:** Indians are not behind anyone, they have contributed remarkable results in fractional calculus. Some of the names are: R. K.Saxena, V. Srivastava, K.N. Rai, S. Das, M. Gupta, S. Saha Roy, V.B.L. Chawrasia, S. K. Mishra, D.K. Upadhyay, P. Prakash, A. Raheem, S. Harikrishnan, S. Abbas, K. Rajagopal, A. Karthikeyan, A. Srinivasan, Jagdev Singh, Amit Goswami, Sajad Jafari.
- **Regional Status:** Many researchers from Maharashtra are actively involved in the study of fractional calculus. Like M. Lolwankar, A. K. Pani, PSV Natraj, S. Bhalekar, V. Daftardar-Gejji, K.C. Takle, J. Verma, D.B. Dhaigude.

6. Scope of Research Work:

- Although a number of researchers have developed new operations and new solutions to fractional diffusion equation and fractional oscillator but still there are several areas of longitudinal oscillations and transverse oscillations which are not covered yet.
- Even many transformations and their applications has been developed by researchers for many problems of science and technology, but an easy, suitable and number of alternate approaches have still has to be put forward. So there is a good scope of development in this area.

7. Expected outcomes:

- We will try to analyse the development in new results and methods for fractional wave equations and fractional oscillator.
- We will try to represent the obtained solutions in graphical way.
- We will try to find approximate solutions of fractional oscillator by old and new techniques.

8. Purpose of carrying out proposed research work:

The purpose of this research is to develop new methods of fractional differential equations involved in fractional wave equations and fractional oscillators to obtain their solutions. Also, we will represent it graphically. The solution of fractional differential

equations obtained by new methods using mathematical software Mathematica. Also we will study the physical significance of waves and oscillations in fractional form. We will compare the solutions obtained by various methods**.**

9. Plan of Work:

Sr. No.	Period	Plan of Work	Targets to be Achieved
1.	First Year	Study of background information relevant to the research issue, such as harmonic oscillators, fractional oscillators, sound waves etc.	Excellent understanding of the subject and well organized course work.
2.	Second Year	Collecting and analysing research articles. And work on some problems which I wish to include in the thesis.	Attempt to obtain fresh results based on existing work and Submit at least three papers.
3.	Third Year	Publishing Some Papers and submitting the thesis.	Some more papers and Thesis submission.

References

[1] V. K. a. J. Lage, "Fractional-Diffusion solutions for transient Local Temperature and heat flux," *ASME J. Heat transfer,* vol. 122, no. 2, pp. 372-376, 2000.

[2] L. Y. a. O. P. Agrawal, A Numerical Scheme for Dynamic Systems Containing Fractional Derivatives, Atlanta, Georgia: Proceedings of DETC'98 ASME Design Engineering Technical Conferences, 1998.

[3] L. S. a. A. Shokooh, "An Eigenvector expansion method for the solution of motion containing fractional derivatives," *Applied Mechanics,* vol. 64, pp. 629-635, 1997.

[4] R. B. a. P. Trovic, "Fractional Calculus- A different approach to the Analysis of Viscoelasticaly Damped Structures," *AIAA Journal,* vol. 21, no. 5, pp. 741-748, 1983.

[5] P. K. a. S. K. L. Gaul, "Impulse response function of an Oscillator with fractional derivative in damped description," *Mechanics research communications,* vol. 16, no. 5, pp. 279-305, 1989.

[6] T. H. a. C. Lorenzo, "A solution to the fundamental linear order differential equations," *NASA TP-1998-208693,* 1998.

[7] R. Koeler, "Applications of Fractional Calculus to the theory of Viscoelasticity," *Applied Mechanics,* vol. 51, pp. 299-307, 1984.

[8] J. F. N. F. a. M. W. K. Diethelm, "Pitfalls in Fast Numerical Solvers for Fractional Equations," *Computer Applications in Mathematics,* vol. 186, no. 2, pp. 482-503, 2006.

[9] Y. Luchko, "Fractional wave equation and damped waves," *Journal of Mathematical Physics,* p. 54, 2013.

[10] L. J., "Memoire sur le calcul des differentielles a indices quelconques," *f ur reine und angrew,* 1832.

[11] K. O. a. J. Spanier, The Fractional Calculus, New York: Academic Press, 1974.

[12] L. S.F., "traite Du Calcul Differential et du Calcul Integral," *Paris Courcier,* vol. 3, pp. 409-410, 1919.

[13] A. N.H., "Solution de quelques problemes a l'aide d'integrales d'enies," *Oeuvres Completes,* vol. 1, pp. 16-18, 1881.

[14] L. J, "Memoire sur lw calcul des differentielles a indices quelconues," *l'Ecole Roy Polytechn,* pp. 71-162, 1832.

[15] L. J., "Memoire sur quelques de geometric et de mecanique," *l'Ecole Roy Polytechn,* pp. 1-69, 1832.

[16] P. I, Fractional Differential Equations, New York: Academic Press, 1990.

[17] Z. T. Trifce Sandev, Fractional Equations and Models, ostrava: Springer, 2019.

[18] W. Wyss, "The Fractional Diffusion Equation," *Journal of Mathematical Physics,* vol. 27, 1986.

[19] F. Mainardi, "Fractional Relaxation- Oscillation and Fractional Diffusion- Wave Phenomena," *Chaos, Solitons and Fractals,* vol. 7, no. 9, pp. 1461-1477, 1996.

[20] R. Hilfer, Applications of Fractional Calculus in Physics, Singapore: World Scientific, 2000.

[21] Y. R. a. A. Puzenko, "Damped oscillations in view of the fractional oscillator equation," *Physical review,* 2002.

[22] A. Tofighi, "The intrinsic damping of the fractional oscillator," *Physica A,* pp. 29-34, 2003.

[23] A. A. Stanislavsky, "Fractional Oscillator," *Physical Review,* vol. 70, 2004.

[24] V. S. S. M. Al Rabtah, "Solutions of a fractional oscillator by using differential transform method," *Elsevier,* vol. 59, pp. 1356-1362, 2010.

[25] J.-S. Duan, "The periodic solution of fractional oscillator equation with periodic input," *Hindawi Publishing Corporation,* 2013.

[26] M. Li, "Three classes of Fractional Oscillators," *Symmetry MDPI,* 2018.

[27] J. H. He, "The simpler, the better: Analytical methods for nonlinear oscillators and fractional oscillators," *Journal of Low frequency noise, vibration and active control,* vol. 38, pp. 1252-1260, 2019.

[28] H. A.-R. G.M. Ismail, "An analytic solution for fractional oscillator in a resisting medium," *Elsevier,* 2019.

[29] R. Parovik, "Mathematical modeling of Linear Fractional Oscillators," *MDPI,* 2020.

[30] X. L. Zhi Yan, "Fractional order harmonic resonance in a multi-frequency excited fractional duffing oscillator with distributed time delay," *eslevier,* 2021.

[31] H. W. QiuBao Wang, "The effect of fractional damping and time-delayed feedback on the stochastic resonance of asymmetric SD oscillator," *Jounal of Latex Templates,* 2022.

[32] G. H. P. a. C. Gerhardt, "Leibnizens gesammelte Werke, Lebinizens mathematische Schriften, Erste Abtheilung," pp. 301,302, 1849.

Miss. Mangrulkar Bhagyashri Umesh
Research Scholar

Dr. Muley Yogesh Manohar
Research Guide